

Colored noise in the fractional Hall effect: duality relations and exact results

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We study noise in the problem of tunneling between fractional quantum Hall edge states within a four probe geometry. We explore the implications of the strong-weak coupling duality symmetry existent in this problem for relating the various density-density auto-correlations and cross-correlations between the four terminals. We identify correlations that transform as either “odd” or “anti-symmetric”, or “even” or “symmetric” quantities under duality. We show that the low frequency noise is colored, and that the deviations from white noise are exactly related to the differential conductance. We show explicitly that the relationship between the slope of the low frequency noise spectrum and the differential conductance follows from an identity that holds to *all* orders in perturbation theory, supporting the results implied by the duality symmetry. This generalizes the results of quantum suppression of the finite frequency noise spectrum to Luttinger liquids and fractional statistics quasiparticles.

PACS: 73.40.Hm, 71.10.Pm, 73.40.Gk

I. INTRODUCTION

Measurements of current fluctuations in a system can yield much information about its excitation spectrum. This has been shown to be the case for tunneling between edge states of fractional quantum Hall (FQH) liquids. Two experimental groups, one in Saclay [1] and another at the Weizmann Institute [2], have recently been able to measure the shot noise level in the tunneling current between FQH edges. The results of the experiments are consistent with the interpretation of tunneling of fractionally charged quasiparticles. The geometries for such measurements are shown in Fig. 1, and studies of various properties of the noise spectrum have been carried out recently [3–9].

For a small tunneling current I_t between the FQH liquid edges, the shot noise level should approach the classical limit $2e^*I_t$, where e^* is either the Laughlin quasiparticle charge ($e^* = \nu e$) for the geometry in Fig. 1a, or the electron charge ($e^* = e$) for the geometry in Fig. 1b [3,4,7]. In this classical limit, the tunneling events are uncorrelated and the noise spectrum at low frequencies appear to be white or frequency independent. As the tunneling current increases, two different effects become manifest. First, the zero-frequency level deviates from the classical level [4,5,9]. Secondly, the low frequency spectrum is no longer white, and develops a cusp at zero frequency [6,7]. This colored noise structure offers the possibility to investigate how the results known for non-interacting particles with Fermi statistics are modified by the correlation effects in FQH liquids, which contain excitations with fractional charge and statistics.

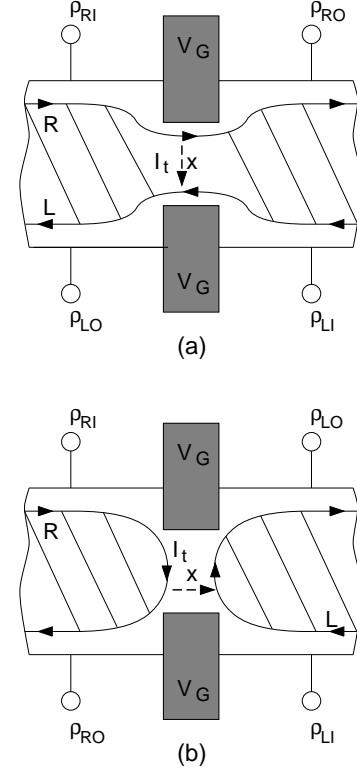


FIG. 1. Two geometries for tunneling between edge states. In (a) quasiparticles can tunnel from one edge to the other. In (b) only electrons can tunnel across. The subscripts for the densities ρ in the four terminals determine the chirality (R, L) of the branch and whether the branch is incoming to (I) or outgoing from (O) the tunneling point.

It has been known for a while that statistics play a role in the suppression of shot noise. In the case of transmission of electrons through a quantum point-contact (QPC), with transmission coefficient T , the zero frequency shot noise level is given by $2eI(1 - T)$, where

$I = \frac{e^2}{h} T V$ is the current transmitted across the point contact. The effects of the fermionic statistics is to reduce by a factor $1 - T$ the classical shot noise level $2eI$ [10]. The implications of quantum statistics, however, are not limited solely to the zero frequency noise level. The noise spectrum is not white, so that the zero-frequency noise level alone cannot describe the full frequency dependent noise spectrum.

For non-interacting electrons, the excess noise spectrum $S_{\text{ex}}(\omega)$, defined as the difference between the non-equilibrium ($V \neq 0$) and equilibrium ($V = 0$) noise, is given by

$$S_{\text{ex}}(\omega) = \frac{e^2}{\pi} T(1 - T) (\omega_J - |\omega|) \theta(\omega_J - |\omega|), \quad (1)$$

where $\omega_J = eV/\hbar$ is the “Josephson” frequency set by the applied voltage. Notice that the excess noise decreases linearly with frequency from the zero-frequency shot noise level $S_{\text{ex}}(0) = 2eI(1 - T)$ to zero at the Josephson frequency ω_J , remaining zero beyond this frequency scale. Although it is hard to probe experimentally the noise spectrum both at high-frequencies near ω_J and at low-frequencies (due to $1/f$ noise), it is possible to observe the spectrum at intermediate frequencies. Indeed, measurements of the spectrum in this intermediate range have been recently obtained by Reznikov *et al.* [11]. These measurements can be extrapolated to zero frequency, yielding the experimental observation of the quantum suppression of shot noise in a QPC by a factor $1 - T$. Hopefully, further refinements of the experimental technique would allow the measurement of the slope $\frac{\Delta S_{\text{ex}}}{\Delta \omega} = -\frac{e^2}{\pi} T(1 - T)$ of the excess noise spectrum, thus probing quantum effects for finite frequencies and showing that the excess noise spectrum is not white.

It is noteworthy that the slope of the noise spectrum near zero frequency keeps a close relationship to the transport coefficient T . This relationship is *not* the one supplied by the fluctuation dissipation theorem (notice that the slope dependence on T is quadratic). One can also find the relationship between T and the slope of the full noise spectrum $S(\omega) = S_{\text{ex}}(\omega) + S_{\text{eq}}(\omega)$ (excess plus equilibrium contribution to noise); using $S_{\text{eq}}(\omega) = \frac{e^2}{\pi} T |\omega|$, one finds that

$$\frac{\Delta S}{\Delta \omega} = \frac{e^2}{\pi} T^2. \quad (2)$$

The relation connecting the slope of the noise spectrum at low-frequencies to the transport coefficient T through the point contact, shown above for non-interacting electrons, can be generalized. In this paper we will generalize such a relation to the case of tunneling between chiral Luttinger liquids. These strongly correlated states have excitations that carry fractional quantum numbers, such as charge and statistics, which make them ideal candidates for the study of the effects on quantum noise due to generalized charge and statistics.

Chiral Luttinger liquids are realized on the edges of fractional quantum Hall liquids. We will study the low-frequency slope of the noise spectrum for a four probe geometry. This consists of looking at density fluctuations in the left (L) and right (R) edges, both incoming (I) to and outgoing (O) from the tunneling point. We will show that correlations between densities in pairs of terminals are related to the transmission and backscattering differential conductances in ways dictated by the strong-weak duality symmetry present in the problem.

The paper is organized as follows. In Section II we summarize our results for the relationships between the slope of the noise spectra and the differential conductances. In Section III we discuss the four terminal geometry for measurement of auto and cross-correlations between pairs of terminals, and we derive in detail the relationships for the correlations of voltage/current fluctuations between pairs of terminals, which follow from the dual descriptions of the problem in terms of electron or quasiparticle tunneling (Fig. 1). By using current conservation and another symmetry operation, which exchanges right R and left L edges and reverses the voltage, we are able to relate correlations in one picture to those in the dual. We then show how these relations tie, in particular, the slope of the spectra to differential conductances in the problem. We derive these relationships between noise spectra slope and differential conductances directly from the boundary sine-Gordon model that describes the tunneling problem in Sections IV (for auto-correlations) and V (for the tunneling current). We show that the relations hold to *all* orders in perturbation theory, signaling the existence of an exact identity. We conclude the paper in Section VI with a discussion of our results, and a comparison to results on the low frequency spectrum obtained from the thermal Bethe ansatz and form factors. In the first reading of the paper, we suggest that readers peruse Section VI prior to Sections IV and V.

II. SUMMARY OF RESULTS

In this section we will give a brief summary of some of the main results and ideas in this paper. We begin by describing the low-frequency structure of the noise spectrum for the tunneling or backscattering current I_t between the edges of the FQH liquid, as depicted in Fig. 2.

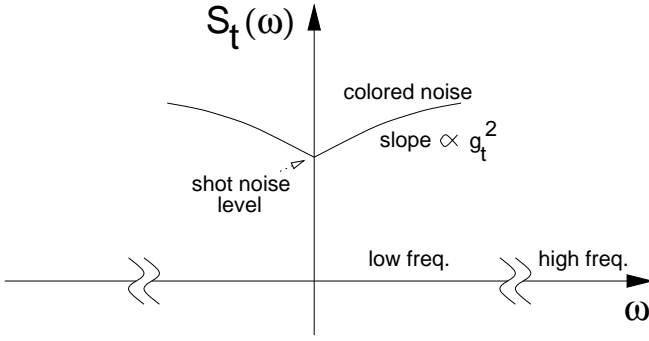


FIG. 2. Low frequency noise spectrum for the tunneling current I_t between edges.

The noise in the backscattering current is defined as

$$S_t(\omega) = \int dt \cos(\omega t) \langle \{I_t(t), I_t(0)\} \rangle .$$

So far, much of the theoretical study of noise in the fractional quantum Hall effect has focused on the zero-frequency shot noise level or $S_t(\omega = 0)$. For small tunneling currents the shot noise approaches the classical level $S_t(\omega = 0) = 2e^* I_t$. The solution for the zero-frequency shot noise is known exactly, via the Bethe ansatz, for any value of the applied voltage V and tunneling amplitude Γ . However, the finite ω results are rather more complicated, and require knowledge of form factors in order to calculate the current-current correlations defining the noise spectrum [6]. In this paper we introduce an alternative approach.

The spectrum at low frequencies is not white, *i.e.*, frequency independent or flat. The frequency dependence or color of the noise can be described in terms of the slope or derivative of the noise spectrum with respect to the frequency:

$$A_t = \frac{1}{G_H} \lim_{\omega \rightarrow 0} \frac{S_t(\omega) - S_t(0)}{\hbar|\omega|} , \quad (3)$$

where we have divided the slope by the Hall conductance $G_H = \nu e^2/h$ in order to make A_t a non-dimensional quantity.

In this paper we will show that the slope A_t is directly related to the differential backscattering conductance $G_t = dI_t/dV$ by

$$A_t = 2 \left(\frac{G_t}{G_H} \right)^2 = 2 g_t^2 , \quad (4)$$

where we also define a dimensionless differential conductance g_t . The relation in Eq. (4) is a generalization of the result for non-interacting electrons (Luttinger parameter $g = 1$) to correlated chiral Luttinger liquids. Similarly, we can relate the noise in the transmission current, $S_T(\omega)$, to the differential transmission conductance $G_T = dI_T/dV$ (with $g_T = G_T/G_H$):

$$A_T = 2 g_T^2 . \quad (5)$$

These results are valid in both the picture in Fig. 1a, where quasiparticles tunnel, or in the picture in Fig. 1b, where electrons tunnel. We indicate which picture we use by a q or e superscript in the quantity of interest, for example, A_t^q, A_t^e and g_t^q, g_t^e .

This brings into focus another point we discuss in this paper: how different quantities transform between the quasiparticle and electron tunneling pictures. For example, it is widely known that

$$g_t^e = 1 - g_t^q , \quad (6)$$

which can be interpreted as a duality relation. This comes about because the two pictures describe the same physical system, but one is the strong tunneling limit of the other and *vice-versa*. Eq. (6) then simply follows from the definitions of I_t and I_T in the two pictures of Fig. 1. These two pictures are actually dual in another, stronger, sense: the model for the two pictures is self-dual, *i.e.*, the tunneling mechanism for the two pictures is described by the same Lagrangian, but with a different sets of parameters (up to counterterms required for renormalization). The parameters are the tunneling amplitude Γ and the Luttinger parameter g . In the quasiparticle tunneling picture they take the values Γ_q and g , and in the electron tunneling picture they take the values Γ_q and g^{-1} , respectively. This allows us to write the conductances in the two pictures using a single function g_t :

$$g_t^q = g_t(g, \Gamma_q) \quad (7)$$

$$g_t^e = g_t(g^{-1}, \Gamma_e) . \quad (8)$$

Because the backscattering conductances g_t^q and g_t^e in the two pictures are related to each other by Eq. (6) in such a simple way, we can ask whether other physical quantities in the two pictures have similar relationships. The coefficients A_t^q and A_t^e do not transform in a such a simple way. Instead, we will show that other quantities, such as the four terminal noise correlations, do have simple transformation laws, although they do not necessarily satisfy the same transformation rules as the conductance. Under the duality transformation, cross-correlations between an incoming and out-going branch transform in an “odd” or “antisymmetric” way like the backscattering conductance. In contrast, the auto-correlations transform in an “even” or “symmetric” way. Specifically, we show that the derivative with respect to frequency of these noise correlations (as defined below) satisfy

$$A_{\text{cross}}^q = 1 - A_{\text{cross}}^e \quad A_{\text{auto}}^q = A_{\text{auto}}^e . \quad (9)$$

We point out that A_t and A_T have both even and odd components, which is why they do not transform in a simple way.

It is tempting to infer from Eqs. (6) and (9) the ansatz $A_{\text{cross}} = g_t$ or $A_{\text{cross}} = g_T$. (There are two combinations of cross correlations between incoming and outgoing branches, depending on whether their chirality is the same or opposite). These ansatz satisfy the correct transformation under duality. Indeed, we show in this paper that this ansatz is correct. This is supported by a perturbative calculation to *all* orders. The calculation is done explicitly using the self-dual boundary sine-Gordon theory that describes the tunneling mechanism in both the quasiparticle and electron tunneling pictures.

The results for the slopes A_{cross}, A_t and A_T can all be expressed directly in terms of the differential conductance, bypassing the non-linear series expansion in terms of the voltage V and the tunneling amplitude Γ . This is also the case for A_{auto} , which we find to be $A_{\text{auto}} = 1 - 2g_t g_T = 1 - 2g_t(1 - g_t)$. In general, we can always write a quantity like A_{cross} as a function $f(g, g_t)$ of both the Luttinger parameter g and the differential conductance g_t . However, our results indicate that the differential conductance g_t is the single parameter controlling the behavior of the slope of the noise.

III. DUALITY RELATIONS

In this section, we will show how the symmetries of the system, which include a voltage reversal symmetry and a duality symmetry, combined with current conservation, lead to identities among the different correlators. We then make use of these identities and one further ansatz about the dependence of the noise on the differential conductance to solve for the noise in the Hall and tunneling currents in terms of differential conductances.

We begin by describing two dual pictures of the system. In the first picture, the constriction is not pinched off, as in Figure 1a, so the quasiparticles can tunnel from one edge to the other. The charge of the particle that tunnels is e^* . Its tunneling amplitude is Γ_q , and the Luttinger parameter is g . One may view g as a parameter controlling the influence of a tunneling event on subsequent ones. In the second picture, the constriction is completely pinched off, so there is no longer any quantum Hall liquid in the central region. Now only electrons can tunnel from one edge to the other, so in this picture the charge of the particle that tunnels is e . Its tunneling amplitude is Γ_e , and the Luttinger parameter is g^{-1} .

There are two ways in which these pictures are dual to one another. We will begin by describing the first one here, and save the second until later in this section. The two pictures are dual to one another in the sense that the strong tunneling limit of one should also describe the weak tunneling limit of the other. For example, as the constriction in Figure 1a is narrowed, it becomes easier for quasiparticles to tunnel from one edge to another, so Γ_q increases. As the constriction is narrowed further (so Γ_q is increased some more) at some point it pinches off

completely, and we obtain the second picture, Figure 1b, which is described by electron tunneling with small Γ_e . Thus the large Γ_q limit of the quasiparticle picture should be the same as the small Γ_e limit of the electron picture, and *vice versa*. In other words, the two pictures should both describe the same physical system.

This means that the incoming and outgoing, right and left moving densities in the two pictures of Fig. 1 are related by

$$\rho_{RI}^q = \rho_{RI}^e \quad (10)$$

$$\rho_{LI}^q = \rho_{LI}^e \quad (11)$$

$$\rho_{RO}^q = \rho_{LO}^e \quad (12)$$

$$\rho_{LO}^q = \rho_{RO}^e, \quad (13)$$

where the subscripts I, O denote incoming and outgoing branches, and R, L denote right-moving and left-moving. The superscript q or e means the function for the density is given in the quasiparticle or electron picture, respectively (see Figure 1). As described above, the densities in the quasiparticle picture are functions of g and Γ_q and the densities in the electron picture are functions of g^{-1} and Γ_e .

These equations can be used to relate the currents and correlations in the quasiparticle picture to those in the electron picture. In order to solve for these correlations, we need further constraints on them. These constraints will take the form of a relation between the correlator in one picture as a function of one set of parameters and the same correlator in that picture as a function of another set of parameters. We will find that these relations will greatly restrict the form of the correlators, but will not determine them uniquely.

To derive these relations, we must make use of additional properties of the system. The first one is current conservation, which simply states that, in a given picture,

$$\rho_{RO} + \rho_{LO} = \rho_{RI} + \rho_{LI}. \quad (14)$$

(Strictly speaking, except right at the impurity, this equation is purely classical. Once these densities appear in expectation values for the noise, the correlators are modified by additional phases $e^{i\omega x}$, where x is the distance along the edge between the terminal in question and the impurity). The second property is that quantities that are quadratic in the densities, such as the noise, are symmetric under inversion of applied voltage

$$V \rightarrow -V.$$

This implies that we can exchange the labels R and L without changing the value of the density-density correlations. To see this, just rotate the sample by 180 degrees, and invert the voltage: these symmetry operations exchange the labels R and L .

Next, it is useful to define some of the currents and noise correlators in terms of the densities. The transmission current is given by $I_T = \langle \rho_{RO} - \rho_{LI} \rangle$ and the tunneling or backscattering current is given by $I_t = \langle \rho_{RI} - \rho_{RO} \rangle$.

It is also useful to define $I_H = \langle \rho_{RI} - \rho_{LI} \rangle$, which is the Hall current in the absence of tunneling or backscattering.

For the noise, we will define, for example,

$$S_{RI,LO}(\omega) = \int dt \cos(\omega t) \langle \{ \rho_{RI}(t), \rho_{LO}(0) \} \rangle \quad (15)$$

The other noise correlators, for instance $S_{RO,LO}$ and $S_{RO,RO}$, are defined likewise. (In general, we will denote by $S_{\alpha,\beta}$ the correlator between the terminals α, β , where α and β can take on the values RO, LO, RI, LI). Notice that we dropped the x dependence of the correlators. We do so because we are primarily interested in the spectrum at low frequencies, in which case the x dependence can be neglected as long as $\omega \ll x^{-1}$. If necessary, to distinguish which picture we are considering we will use the superscripts e and q .

We define the noise $S^{(0)}$ as the correlator between two right or two left moving densities in the absence of the coupling Γ between the R and L branches in a particular picture. It is given by

$$S^{(0)}(\omega) = \frac{\nu}{2\pi} |\omega|.$$

We note that $S_{LI,LI} = S_{RI,RI} = S^{(0)}$ in the presence of any tunneling because the incoming channels have yet to be affected by the tunneling. Similarly, $S_{LI,RI} = S_{RI,LI} = 0$ since the two incoming channels are completely uncorrelated.

It follows from current conservation and the symmetry under voltage inversion ($R \leftrightarrow L$) that, in a given picture,

$$I_T = I_H - I_t \quad (16)$$

$$S_{RI,RO} = S_{LI,LO} = S^{(0)} - S_{RI,LO} = S^{(0)} - S_{LI,RO} \quad (17)$$

$$S_{RO,RO} = S_{LO,LO} = S^{(0)} - S_{RO,LO}. \quad (18)$$

These equations relate one set of currents or noise in one picture to another set of currents or noise in the same picture. Next, we can use the relations between the two pictures to write expressions for the current or noise in one picture in terms of the same current or noise found in the other picture. Combining Eqs. (16-18) and (10-13), we find

$$I_t^q = I_H - I_t^e \quad (19)$$

$$S_{RI,RI}^e = S_{RI,RI}^q = S^0 = \frac{\nu}{2\pi} |\omega| \quad (20)$$

$$S^{(0)} - S_{RI,LO}^q = S_{RI,LO}^e \quad (21)$$

$$S_{RO,LO}^q = S_{RO,LO}^e. \quad (22)$$

Because these equations relate the noise in one picture to the noise in the other picture, we cannot make further use of these equations unless we either know what the noise is in one of the two pictures, or we know another relation between the noise in the two different pictures. If we make an additional assumption, we can obtain this second set of relations.

In particular, there is a second sense in which we mean the two pictures are dual: we assume that the tunneling is described by a self-dual theory, by which we mean that the description of the tunneling mechanism is the same in both pictures. Pictorially, what this means is that we can reverse the shaded and unshaded regions in Figure 1b so that it looks exactly like Figure 1a, rotated by 90 degrees. This signifies that now in both pictures the tunneling should be described in the same way, just with differing parameters – g^{-1} , e , and Γ_e , or g , e^* and Γ_q – depending on whether quasiparticles or electrons are tunneling. (However, the filling fraction, ν , of the shaded region remains the same in both cases.) By using the Luttinger liquid framework for both the electron tunneling of Figure 1b and the quasiparticle tunneling of Figure 1a, we are implicitly making this assumption. (However, once questions of renormalization arise and counter-terms must be added to one picture and not the other, it is no longer guaranteed that the system really is described by a self-dual theory.) Mathematically, in this framework we use the same Lagrangian, just with different values of charge and tunneling amplitude. Given this second type of duality, we can replace $S_{\alpha,\beta}^e$ and $S_{\alpha,\beta}^q$ by a single function, $S_{\alpha,\beta}(g, \Gamma)$, so that $S_{\alpha,\beta}^e = S_{\alpha,\beta}(g^{-1}, \Gamma_e)$ and $S_{\alpha,\beta}^q = S_{\alpha,\beta}(g, \Gamma_q)$, (and similarly for the currents). The identities between the noise and currents then become

$$I_t(g, \Gamma_q) = I_H - I_t(g^{-1}, \Gamma_e) \quad (23)$$

$$S^{(0)} - S_{RI,LO}(g, \Gamma_q) = S_{RI,LO}(g^{-1}, \Gamma_e) \quad (24)$$

$$S_{RO,LO}(g, \Gamma_q) = S_{RO,LO}(g^{-1}, \Gamma_e). \quad (25)$$

Now each of these equations is a duality relation relating a function at one set of parameters to the same function at another set of parameters. This kind of relation greatly restricts the possible form of the current and noise.

To find the form of the noise that is suggested by these duality relations, we begin by noting that Eq. (23) for I_t and Eq. (25) for $S_{RI,LO}$ have a very similar form, which indicates there may be a simple relation between the function that satisfies the duality relation for the current and the one that satisfies the duality relation for the cross-correlation $S_{RI,LO}$. One must take care, though, in trying to equate I_t and $S_{RI,LO}$ because they have different dimensions and one is a function of ω and the other is not. Instead, we will look at the slope $A_{RI,LO}$ of $S_{RI,LO}$ near $\omega = 0$, which we will normalize as follows:

$$A_{RI,LO} = \lim_{\omega \rightarrow 0} \frac{S_{RI,LO}^{\text{sing}}}{S^{(0)}}. \quad (26)$$

In this equation, $S_{RI,LO}^{\text{sing}}$ is the part of $S_{RI,LO}$ that is singular as $\omega \rightarrow 0$. We will also define the dimensionless differential conductance as

$$g_t = \frac{1}{G_H} \frac{dI_t}{dV}, \quad (27)$$

where $G_H = \nu e^2/h$. Then the duality relations for $A_{RI,LO}$ and g_t become

$$g_t(g, \Gamma_q) = 1 - g_t(g^{-1}, \Gamma_e) \quad (28)$$

$$A_{RI,LO}(g, \Gamma_q) = 1 - A_{RI,LO}(g^{-1}, \Gamma_e). \quad (29)$$

Thus the dimensionless conductance and the normalized slope of the noise satisfy exactly the same duality relation.

To look for a simple relation between the conductance and $A_{RI,LO}$, we consider the case when the Luttinger parameter $g = 1/2$. In this case, there is an exact solution for the noise. Guided by the fact that the conductance and the slope of the noise satisfy the same duality relation, we find we can rewrite the expression for $A_{RI,LO}$, calculated in reference [8], in terms of the conductance. It is given by

$$A_{RI,LO} = g_t. \quad (30)$$

Thus we see that in this case there is, indeed, a simple relation between $A_{RI,LO}$ and the differential conductance, which begs the question of whether this is true for all g . In Section IV, we will explicitly calculate the slope of the noise to all orders in perturbation theory. This perturbative expansion is valid when $(V/T_B)^{(2g-2)}$ is small, where $T_B \propto \Gamma^{-1/(g-1)}$. However, if we want to know the value for $(V/T_B)^{(2g-2)}$ large, we can use the duality relation and calculate in the dual picture. In this way, we obtain the value of $A_{RI,LO}$ over the whole region of parameter space. We find that Equation (30) does hold for all g .

Here, instead, we will show how relation (30) follows from one additional ansatz. Coloumb gas expansions for the conductance and noise are both power series in $(V/T_B)^{(2g-2)}$ beginning with the term $(V/T_B)^{(2g-2)}$. As a consequence, we can always express $A_{RI,LO}$ as a power series in g_t , given by

$$A_{RI,LO} = \sum_{n=1}^{\infty} a_n(g) g_t^n, \quad (31)$$

where $a_n(g)$ depends on the Luttinger parameter g . Because both $A_{RI,LO}$ and g_t satisfy the duality relation, this puts some restrictions on the a_n , but does not determine them uniquely. However, if we make the assumption that a_n does not depend on g , then we can use the solution at $g = 1/2$ to fix the a_n , with the result that $A_{RI,LO} = g_t$ for all g . (This assumption is equivalent to the one made in [9] that enabled Weiss to use the duality relation to solve for the conductance. Since $A_{RI,LO}$ satisfies the same duality relation as the conductance and equals the conductance for $g = 1/2$, his ansatz and ensuing calculation also uniquely determine $A_{RI,LO}$).

We can use the same line of reasoning for the slope of $S_{RO,LO}$, which is defined as

$$A_{RO,LO} = \lim_{\omega \rightarrow 0} \frac{S_{RO,LO}^{\text{sing}}}{S^{(0)}}. \quad (32)$$

For $g = 1/2$, this can be written in terms of the differential conductance as follows:

$$A_{RO,LO} = 2(g_t - g_t^2). \quad (33)$$

Given the duality relation for g_t , this expression satisfies the duality relation for $A_{RO,LO}$. As before, $A_{RO,LO}$ is a power series in $(V/T_B)^{(2g-2)}$ starting with the first order term, so it is given by

$$A_{RO,LO} = \sum_{n=1}^{\infty} b_n(g) g_t^n. \quad (34)$$

If again we assume the b_n are independent of g , then the value of $A_{RO,LO}$ at $g = 1/2$ determines b_n for all g , and the slope of the noise has the form given in Eq. (33). Combining this result with Eq. (18), which relates correlations between pairs of outgoing branches, we can write $A_{RO,RO} = 1 - A_{RO,LO}$, or

$$A_{\text{auto}} = A_{RO,RO} = 1 - 2g_t g_T. \quad (35)$$

This is our conjecture for the auto correlations in the outgoing branches.

Finally, we conclude this section by stating the conjecture for the slope of the noise in the tunneling current, A_t , and in the transmission current, A_T . The slope of the tunneling noise is defined as

$$A_t = \lim_{\omega \rightarrow 0} \frac{S_t}{S^{(0)}}, \quad (36)$$

where S_t^{sing} is the singular part of the tunneling noise, and A_T is defined similarly. Using the definition of the transmission and tunneling currents, we find

$$A_t = -A_{RO,LO} + 2A_{RI,LO} \quad (37)$$

and

$$A_T = 2 - A_{RO,LO} - 2A_{RI,LO} \quad (38)$$

The noise in the transmission current in one picture equals the noise in the tunneling current in the other picture. However, in a given picture, there is no simple duality relation for A_t or A_T . Instead, our solutions for $A_{RO,LO}$ and $A_{RI,LO}$ imply that

$$A_t = 2g_t^2, \quad (39)$$

$$A_T = 2g_T^2. \quad (40)$$

In Section V, we use a multipole expansion to calculate the tunneling noise to all orders in perturbation theory, and find that Equation (40) does, indeed, hold. Thus, we expect to find a simple relation between the slope of the noise at low frequency and the square of the differential conductance. This has the same form as Shiba's relation for the dissipative two-level system [12,13].

IV. CROSS-CORRELATIONS

In the previous section we related the slopes of different noise correlations to the differential conductances by exploring the duality symmetry in the problem. We used an ansatz that the coefficients for the series expansions of these slopes (as a power series in the conductances) did not depend on the Luttinger parameter. In this section we formally justify this ansatz by explicitly calculating, starting from the boundary sine-Gordon Lagrangian, the noise cross-correlations between an incoming branch and an outgoing branch. In other words, here we show that the independence of the expansion coefficients on the Luttinger parameter g is a specific property of the boundary sine-Gordon Lagrangian that describes the tunneling problem.

The Lagrangian that describes the tunneling between chiral Luttinger liquids through a QPC is

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_L + \Gamma \delta(x) e^{i\omega_J t} e^{i\sqrt{g}(\phi_R(t,0) + \phi_L(t,0))} , \quad (41)$$

where $\mathcal{L}_{R,L} = \frac{1}{4\pi} \partial_x \phi_{R,L} (\mp \partial_t - \partial_x) \phi_{R,L}$ is the Lagrangian for the free chiral bosons. We will calculate the zero-frequency singularity in the noise spectrum to all orders in a perturbative expansion in Γ .

We will show that the slope of the spectrum for the cross-correlations is linearly related to the differential transmission and backscattering conductances. We proceed in the following way: we will expand the density-density correlations for the cross noise to all orders in the tunneling amplitude Γ , and compare them to the expansion, also to all orders, of the differential conductances.

We write the correlations between density operators as follows:

$$\langle \rho_a(t, x_1) \rho_b(0, x_2) \rangle , \quad (42)$$

where a, b take the values $+1$ for R moving branches and -1 for L moving ones. Such compressed notation makes it simpler to identify incoming and outgoing branches in a unified way for both left and right movers: $\rho_a(t, x_1)$, for example, is the density in an incoming or outgoing branch if $ax_1 < 0$ or $ax_1 > 0$, respectively.

The densities are related to the fields $\phi_{R,L}$ through $\rho_{R,L} = \frac{\sqrt{\nu}}{2\pi} \partial_x \phi_{R,L}$, so that we can write

$$\langle \rho_a(t, x_1) \rho_b(t', x_2) \rangle = \frac{\nu}{(2\pi)^2} \partial_{x_1} \partial_{x_2} \langle \phi_a(t, x_1) \phi_b(t', x_2) \rangle , \quad (43)$$

where it is convenient to use

$$\langle \phi_a(t, x_1) \phi_b(t', x_2) \rangle = \frac{d}{d\lambda_1} \frac{d}{d\lambda_2} \langle e^{i\lambda_1 \phi_a(t, x_1)} e^{-i\lambda_2 \phi_b(t', x_2)} \rangle \Big|_{\lambda_1, \lambda_2=0} . \quad (44)$$

The last correlation function is easy to calculate perturbatively using

$$\langle T_c(e^{i\lambda_1 \phi_a(t, x_1)} e^{-i\lambda_2 \phi_b(t', x_2)}) \rangle = \langle 0 | T_c(S(-\infty, -\infty) e^{i\lambda_1 \phi_a(t, x_1)} e^{-i\lambda_2 \phi_b(t', x_2)}) | 0 \rangle , \quad (45)$$

where $|0\rangle$ is the unperturbed ground state, and T_c is the ordering along the Keldysh contour (see Fig. 3 and Refs. [7,8]). The scattering operator $S(-\infty, -\infty)$ takes the initial state, evolves it from $t = -\infty$ to $t = \infty$ and back to $t = -\infty$. The use of the Keldysh contour is necessary in the treatment of non-equilibrium problems, such as the one we have in hand. A more detailed description of the method in the context treated here can be found in Ref. [7].

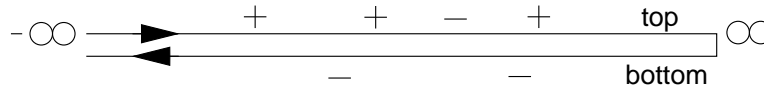


FIG. 3. Keldysh contour for the non-equilibrium Coulomb gas expansion. Time evolves forward in the top part of the contour, and backwards in the bottom. Charges from the Coulomb gas expansion are inserted in both the top and bottom pieces.

The perturbative treatment corresponds to a Coulomb gas expansion [7]. The nonzero contribution to the correlation above comes from the neutral terms in the expansion, thus only even orders in Γ contribute. To $(2n)$ -th order, we have an insertion of n positive charges and n negative charges. We will label the times at which they are inserted in the expansion t_i , $i = 1, \dots, 2n$, with $i = 1, \dots, n$ corresponding to the $+$ charges ($q_i = +1$), and $i = n + 1, \dots, 2n$ for the $-$ charges ($q_i = -1$).

$$\langle T_c(e^{i\lambda_1\phi_a(t,x_1)} e^{-i\lambda_2\phi_b(t',x_2)}) \rangle = \sum_{n=0}^{\infty} (i\Gamma)^n (i\Gamma^*)^n \oint_c \prod_{i=1}^{2n} dt_i e^{i\sum_{i=1}^n \omega_0(t_i - t_{n+i})} \langle 0 | T_c \left(e^{i\lambda_1\phi_a(t,x_1)} e^{-i\lambda_2\phi_b(t',x_2)} \prod_{i=1}^{2n} e^{iq_i \sqrt{g}\phi(t_i,0)} \right) | 0 \rangle, \quad (46)$$

where ϕ without subscript stands for the sum $\phi_R + \phi_L$. The expression above is simplified using

$$\langle 0 | T_c \left(\prod_j e^{iq_j \sqrt{g}\phi(t_j, x_j)} \right) | 0 \rangle = e^{-\frac{g}{2} \sum_{i \neq j} q_i q_j \langle 0 | T_c(\phi(t_i, x_i) \phi(t_j, x_j)) | 0 \rangle}. \quad (47)$$

Substituting it into Eq. (44) we obtain

$$\begin{aligned} \langle T_c(\phi_a(t, x_1) \phi_b(t', x_2)) \rangle &= \sum_{n=0}^{\infty} (-1)^n |\Gamma|^{2n} \oint_c \prod_{i=1}^{2n} dt_i e^{i\sum_{i=1}^n \omega_0(t_i - t_{n+i})} e^{-\frac{g}{2} \sum_{i \neq j} q_i q_j \langle 0 | T_c(\phi(t_i) \phi(t_j, x_j)) | 0 \rangle} \\ &\quad \times \left\{ \left[\sum_{i=1}^{2n} q_i \sqrt{g} \langle 0 | T_c(\phi(t_i, 0) \phi_a(t, x_1)) | 0 \rangle \right] \times \left[\sum_{j=1}^{2n} q_j \sqrt{g} \langle 0 | T_c(\phi(t_j, 0) \phi_b(t', x_2)) | 0 \rangle \right] \right. \\ &\quad \left. + \langle 0 | T_c(\phi_a(t, x_1) \phi_b(t', x_2)) | 0 \rangle \right\} \end{aligned} \quad (48)$$

The last term in the expression above is simply proportional to $\langle 0 | T_c(\phi_a(t, x_1) \phi_b(t', x_2)) | 0 \rangle$. The proportionality constant is equal to $Z = \langle 0 | S(-\infty, -\infty) | 0 \rangle \equiv 1$. This is the zero-order contribution.

In order to carry out the calculations, we introduce notation that keeps track of the position of the inserted charges along the contour, *i.e.*, whether they are in the forward (or top) branch, or in the return (or bottom) branch (see Fig. 3 and Refs. [7,8]). The position of the charges is important for the computation of the contour-ordered correlation function $\langle 0 | T_c(\phi_{R,L}(t_1, x_1) \phi_{R,L}(t_2, x_2)) | 0 \rangle$

$$= \begin{cases} -\ln\{\delta + i \operatorname{sign}(t_1 - t_2)[(t_1 - t_2) \mp (x_1 - x_2)]\}, & \text{both } t_1 \text{ and } t_2 \text{ in the top branch} \\ -\ln\{\delta - i \operatorname{sign}(t_1 - t_2)[(t_1 - t_2) \mp (x_1 - x_2)]\}, & \text{both } t_1 \text{ and } t_2 \text{ in the bottom branch} \\ -\ln\{\delta - i[(t_1 - t_2) \mp (x_1 - x_2)]\}, & t_1 \text{ in the top and } t_2 \text{ in the bottom branch} \\ -\ln\{\delta + i[(t_1 - t_2) \mp (x_1 - x_2)]\}, & t_1 \text{ in the bottom and } t_2 \text{ in the top branch.} \end{cases} \quad (49)$$

The compact notation consists of giving indices to the times which contain the information about which branch of the Keldysh contour they are on, so that t^μ is on the top branch for $\mu = +1$, and on the bottom for $\mu = -1$. In this way, we can compress the correlations to a compact form:

$$\begin{aligned} G_{\mu_1 \mu_2}^{ab}(t_1, x_1; t_2, x_2) &= G_{\mu_1 \mu_2}^{ab}(t_1 - t_2, x_1 - x_2) = \langle 0 | T_c(\phi_a(t^{\mu_1}, x_1) \phi_b(t^{\mu_2}, x_2)) | 0 \rangle \\ &= -\delta_{a,b} \ln(\delta + i K_{\mu_1 \mu_2}(t_1 - t_2)[(t_1 - t_2) - a(x_1 - x_2)]), \end{aligned} \quad (50)$$

where $K_{\pm\pm}(t) = \pm \operatorname{sign}(t)$ and $K_{\pm\mp}(t) = \mp 1$. Again, we have used $a, b = \pm 1$ for R and L fields, respectively. The correlation in Eq. (48) can be written, using this compressed notation, as

$$\begin{aligned} \langle T_c(\phi_a(t, x_1) \phi_b(t', x_2)) \rangle &= G_{++}^{ab}(t - t', x_1 - x_2) + \\ &g \sum_{n=1}^{\infty} (-1)^n |\Gamma|^{2n} \sum_{\{\mu_i\}} \int_{-\infty}^{\infty} \prod_{i=1}^{2n} \mu_i dt_i e^{i\sum_{i=1}^n \omega_0(t_i - t_{n+i})} P_{\mu_1, \dots, \mu_{2n}}(t_1, \dots, t_{2n}) \\ &\quad \times \left[\sum_{i=1}^{2n} q_i G_{+\mu_i}^{aa}(t - t_i, x_1) \right] \times \left[\sum_{j=1}^{2n} q_j G_{+\mu_j}^{bb}(t' - t_j, x_2) \right], \end{aligned} \quad (51)$$

where $P_{\mu_1, \dots, \mu_{2n}}(t_1, \dots, t_{2n}) = e^{-\frac{g}{2} \sum_{i \neq j} q_i q_j [G_{\mu_i \mu_j}^{++}(t_i - t_j, 0) + G_{\mu_i \mu_j}^{--}(t_i - t_j, 0)]}$. The factors μ_i simply keep track of the sign coming from the integration of the times t_i along the contour. Notice that the times t and t' are taken to be on the top branch.

Now, let

$$\begin{aligned} F_{ab}(\omega; x_1, x_2) &= \int_{-\infty}^{\infty} dt e^{i\omega t} \langle T_c(\rho_a(t, x_1) \rho_b(0, x_2)) \rangle \\ &= \frac{\nu}{(2\pi)^2} \partial_{x_1} \partial_{x_2} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle T_c(\phi_a(t, x_1) \phi_b(0, x_2)) \rangle, \end{aligned} \quad (52)$$

which can be easily shown, using Eq. (51), to yield

$$\begin{aligned} F_{ab}(\omega; x_1, x_2) &= -\frac{\nu}{(2\pi)^2} \partial_{x_1} \tilde{g}_{++}^{ab}(\omega, x_1 - x_2) \\ &+ \frac{\nu g}{(2\pi)^2} \sum_{n=1}^{\infty} (-1)^n |\Gamma|^{2n} \sum_{\{\mu_i\}} \int_{-\infty}^{\infty} \prod_{i=1}^{2n} \mu_i dt_i e^{i \sum_{i=1}^n \omega_0(t_i - t_{n+i})} P_{\mu_1, \dots, \mu_{2n}}(t_1, \dots, t_{2n}) \\ &\times \left[\sum_{i=1}^{2n} q_i \tilde{g}_{+\mu_i}^{aa}(\omega, x_1) e^{i\omega t_i} \right] \times \left[\sum_{j=1}^{2n} q_j g_{+\mu_j}^{bb}(t' - t_j, x_2) \right]. \end{aligned} \quad (53)$$

In this equation, the function g is given by $g_{\mu_1 \mu_2}^{ab}(t, x) = \partial_x G_{\mu_1 \mu_2}^{ab}(\omega, x)$ and \tilde{g} is the Fourier transform of g ; they can be obtained from Eq. (50):

$$\tilde{g}_{\mu_1 \mu_2}^{ab}(\omega, x) = \delta_{a,b} \times \begin{cases} \pi i a e^{i\omega a x} (\text{sign}(\omega) + \text{sign}(ax)) & , \mu_1 = +1, \mu_2 = +1 \\ \pi i a e^{i\omega a x} (\text{sign}(\omega) - \text{sign}(ax)) & , \mu_1 = -1, \mu_2 = -1 \\ -2\pi i a e^{i\omega a x} \theta(-\omega) & , \mu_1 = +1, \mu_2 = -1 \\ 2\pi i a e^{i\omega a x} \theta(\omega) & , \mu_1 = -1, \mu_2 = +1 \end{cases} \quad (54)$$

We will now take $ax_1 < 0$ (incoming state) and $bx_2 > 0$ (outgoing state), so as to discuss the cross-correlations. In this case, $\tilde{g}_{+\mu}^{aa}(\omega, x_1) = -2\pi i a e^{i\omega a x_1} \theta(-\omega)$, for both $\mu = \pm 1$. For small $|\omega|$ we have

$$\sum_{i=1}^{2n} q_i \tilde{g}_{+\mu_i}^{aa}(\omega, x_1) e^{i\omega t_i} = -2\pi a \theta(-\omega) \omega \sum_{i=1}^n (t_i - t_{n+i}) + \mathcal{O}(\omega^2). \quad (55)$$

One can thus write

$$\begin{aligned} F_{ab}(\omega; x_1, x_2) &= -\delta_{a,b} \frac{\nu}{(2\pi)} \omega \theta(-\omega) \\ &- a \theta(-\omega) \omega \frac{\nu g}{(2\pi)} \sum_{n=1}^{\infty} (-1)^n |\Gamma|^{2n} \sum_{\{\mu_i\}} \int_{-\infty}^{\infty} \prod_{i=1}^{2n} \mu_i dt_i \sum_{i=1}^n (t_i - t_{n+i}) e^{i \sum_{i=1}^n \omega_0(t_i - t_{n+i})} \\ &\times P_{\mu_1, \dots, \mu_{2n}}(t_1, \dots, t_{2n}) \left[\sum_{j=1}^{2n} q_j g_{+\mu_j}^{bb}(t' - t_j, x_2) \right] + \mathcal{O}(\omega^2). \end{aligned} \quad (56)$$

We can similarly (and more easily) expand the tunneling current (*i.e.*, the difference in the densities in a branch before and after the impurity) to all orders in Γ .

$$\begin{aligned} I_t &= b(\langle \rho_b(t', x_2) \rangle - \langle \rho_b(t', -x_2) \rangle) = -ib \frac{\sqrt{\nu g}}{2\pi} \sum_{n=1}^{\infty} (-1)^n |\Gamma|^{2n} \sum_{\{\mu_i\}} \int_{-\infty}^{\infty} \prod_{i=1}^{2n} \mu_i dt_i e^{i \sum_{i=1}^n \omega_0(t_i - t_{n+i})} \\ &\times P_{\mu_1, \dots, \mu_{2n}}(t_1, \dots, t_{2n}) \left[\sum_{j=1}^{2n} q_j g_{+\mu_j}^{bb}(t' - t_j, x_2) \right]. \end{aligned} \quad (57)$$

The current is defined as positive flowing from the right to the left edge, hence the factor b in the expression above. By direct comparison with Eq. (56), one can then write

$$F_{ab}(\omega; x_1, x_2) = \left(-\delta_{a,b} \frac{\nu}{(2\pi)} + \sqrt{\nu g} ab \frac{dI_t}{d\omega_0} \right) \omega \theta(-\omega) + \mathcal{O}(\omega^2) \quad (58)$$

The noise spectrum is obtained from $F_{ab}(\omega, x_1, x_2)$ as follows:

$$\begin{aligned} S_{ab}(\omega; x_1, x_2) &= S_{ba}(-\omega; x_2, x_1) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \{ \rho_a(t, x_1), \rho_b(0, x_2) \} \rangle \\ &= F_{ab}(\omega; x_1, x_2) + F_{ab}^*(-\omega; x_1, x_2), \end{aligned} \quad (59)$$

so that we can finally write

$$\begin{aligned} S_{ab}(\omega; x_1, x_2) &= \left(\delta_{a,b} \frac{\nu}{(2\pi)} - \sqrt{\nu g} ab \frac{dI_t}{d\omega_0} \right) |\omega| + \mathcal{O}(\omega^2) \\ &= \frac{\nu}{(2\pi)} |\omega| \left(\delta_{a,b} - ab \frac{G_t}{G_H} \right) + \mathcal{O}(\omega^2), \end{aligned} \quad (60)$$

where $G_t = \frac{dI_t}{dV}$, *i. e.*, the differential conductance, and $G_H = \nu e^2/h$ is the quantized Hall conductance. We used above that $\omega_0 = e^*V/\hbar$, and that $e^* = \sqrt{\nu g}$ (in units of $e = 1$). Reinserting back \hbar and e (which were both set to 1), we can write the result in a more physical way:

$$S_{ab}(\omega; x_1, x_2) = \hbar |\omega| (\delta_{a,b} G_H - ab G_t), \quad (61)$$

the final result for cross-correlations (valid when $(ax_1) \times (bx_2) < 0$).

Notice that the result above satisfies the strong-weak coupling duality symmetry for the cross-noise. For example, $S_{RI,RO}^e(\omega) = S_{RI,LO}^q(\omega)$ should be satisfied. Using the result calculated above, $S_{RI,RO}^e(\omega) = \hbar |\omega| (G_H - G_t^e)$, and $S_{RI,LO}^q(\omega) = \hbar |\omega| G_t^q$. Now, $G_H - G_t^e = G_T^e = G_t^q$, so that, indeed, $S_{RI,RO}^e(\omega) = S_{RI,LO}^q(\omega)$.

Finally, we can divide the cross-correlations by the equilibrium noise $S^{(0)}(\omega) = \frac{\nu}{2\pi} |\omega| = \hbar |\omega| G_H$, and cast the result in terms of normalized conductances $g_t = G_t/G_H$ and $g_T = G_T/G_H$:

$$A_{aI,bO} = \delta_{a,b} - ab g_t = \begin{cases} g_T, & a = b \\ g_t, & a \neq b \end{cases}. \quad (62)$$

V. AUTO-CORRELATIONS

In this section, we will complete our calculation of the slope of the low frequency noise by finding the slope of the noise in the tunneling current A_t . From A_t and the cross-correlation calculated the previous section we can find all the other correlators. Since A_t comes from the noise in the tunneling current, we will calculate directly the tunneling current-current correlation, given by

$$S_t(\omega) = \int_{-\infty}^{\infty} dt \cos(\omega t) \langle \{ I_t(t), I_t(0) \} \rangle, \quad (63)$$

where the tunneling operator is given by

$$I_t(t) = i\Gamma e^{i\omega_0 t} e^{i\sqrt{g}\phi(t,0)} - i\Gamma^* e^{-i\omega_0 t} e^{-i\sqrt{g}\phi(t,0)}, \quad (64)$$

and $\phi(t,0) = \phi_R(t,0) + \phi_L(t,0)$. Notice that $S_t(\omega) = S_t(-\omega)$, so that any term in S_t that is linear in ω and analytic must vanish. Thus, at first order, the noise goes as $|\omega|$. Also, because of the translational invariance of the correlator, we can replace $\cos(\omega t)$ by $e^{i\omega t}$.

For comparison, we will once again need the tunneling current $I_t = \langle I_t(t) \rangle$, where $I_t(t)$ is defined in Eq. (64). We can use the expression for the tunneling current to obtain an expansion similar to the one in Section IV. In this case there is one “physical” charge in the Coulomb gas which comes from the operator $I_t(t)$. It is located at time t on the top branch and its charge is q_0 , which can be ± 1 . The remaining inserted charges occur at times t_i and can lie on either the top or bottom branch, labeled by μ_i , with $i = 1, \dots, 2n-1$. They have charges q_i which are chosen so that the total charge (including q_0) is zero. With these definitions, the perturbation series for I_t is

$$I_t = \sum_{n=1}^{\infty} I^{(2n)}, \quad (65)$$

where

$$I^{(2n)} = \frac{(-1)^n |\Gamma|^{2n}}{n!(n-1)!} \sum_{\{\mu_i\}} \sum_{q_0=\pm 1} \int_{-\infty}^{\infty} \prod_{i=1}^{2n-1} \mu_i dt_i e^{i \sum_{i=1}^{2n-1} \omega_0 q_i t_i} e^{i \omega_0 q_0 t} P_{+, \mu_1, \dots, \mu_{2n-1}}(t, t_1, \dots, t_{2n-1}). \quad (66)$$

In this equation, P is defined as in Section IV.

For the noise, the Coulomb gas has a charge q_0 at $t \equiv t_0$ and a charge p_0 at $s \equiv s_0$ which are both on the top branch. Both of these charges can be ± 1 . The remaining charges q_i are at positions t_i which can be on either branch, labeled by μ_i , with $i = 1, \dots, 2n-2$. The sum of all the charges must again be neutral. In particular, this implies that the minimum number of inserted charges $n_{\min}(p_0, q_0)$ in the perturbative expansion is $n_{\min} = 0$ if p_0 and q_0 have opposite signs, and $n_{\min} = 2$ if p_0 and q_0 have the same sign.

The perturbation series for the noise in the tunneling current is then

$$S_t(\omega) = 2 \int_{-\infty}^{\infty} d(t-s) e^{i\omega(t-s)} \sum_{\{\mu_i\}} \sum_{q_0, p_0=\pm 1} \sum_{n=1+\frac{n_{\min}}{2}}^{\infty} \frac{(-1)^n |\Gamma|^{2n}}{n_+! n_-!} \int_{-\infty}^{\infty} \prod_{i=1}^{2n-2} \mu_i dt_i e^{i \sum_{i=1}^{2n-2} \omega_0 q_i t_i} e^{i \omega_0 (q_0 t + p_0 s)} \times P_{+, +, \mu_1, \dots, \mu_{2n-2}}(t, s, t_1, \dots, t_{2n-2}), \quad (67)$$

where n_+ is the number of inserted charges that are positive and n_- is the number of negative inserted charges. The factor of 2 in front of the whole expression accounts for expanding the anti-commutator in the definition of the noise into a time-ordered and an anti-time-ordered piece; for the $|\omega|$ singularity, both contribute the same, hence we work with only the time-ordered and include the factor of 2.

When $g > 1$ and V is small, and also when $g < 1$ and V is large, the leading contribution to the singularities should come from the configurations where each of the charges at the t_i 's are close to either the charge at t or the charge at s . For a particular configuration, we will let t_i for $i = 1, \dots, m$ be the coordinates of the charges close to t , and s_i for $i = 1, \dots, m'$ be the coordinates of the charges close to s . The charge at time t_i has charge q_i and is on the branch labeled by μ_i . Similarly, the charge at s_i has charge p_i and is on the branch ν_i . Then

$$P_{+, +, \mu_1, \dots, \mu_{2n-2}}(t, s, t_1, \dots, t_{2n-2}) = P_{+, \mu_1, \dots, \mu_m, +, \nu_1, \dots, \nu_{m'}}(t, t_1, \dots, t_m, s, s_1, \dots, s_{m'}) \quad (68)$$

$$= P_{+, \mu_1, \dots, \mu_m}(t, t_1, \dots, t_m) P_{+, \nu_1, \dots, \nu_{m'}}(s, s_1, \dots, s_{m'}) \times R(t, t_1, \dots, t_m, s, s_1, \dots, s'_{m'}), \quad (69)$$

where

$$R(t, t_1, \dots, t_m, s, s_1, \dots, s'_{m'}) = e^{-g \sum_{i,j} q_i p_j [G_{\mu_i \nu_j}^{++}(t_i - s_j, 0) + G_{\mu_i \nu_j}^{--}(t_i - s_j, 0)]}. \quad (70)$$

R contains all the interactions between the charges in one multipole and the charges in the other. With the definition of the propagators G , the expression for R becomes

$$R(t, t_1, \dots, t_m, s, s_1, \dots, s'_{m'}) = \prod_{i=0}^m \prod_{j=0}^{m'} (\delta + i K_{\mu_i \nu_j}(t_i - s_j))^{2q_i p_j g}. \quad (71)$$

If all the t_i are shifted by t and all the s_i are shifted by s , then the expression for $S(\omega)$ becomes

$$2 \sum_{q_0, p_0=\pm 1} \sum_{m, m'=1}^{\infty} \int_{-\infty}^{\infty} d(t-s) e^{i\omega(t-s)} \tilde{I}_t^{(m)} \tilde{I}_t^{(m')} R(t, t_1, \dots, t_m, s, s_1, \dots, s_m) e^{iQ\omega_0(t-s)}, \quad (72)$$

where Q is the total charge in the multipole around t . $\tilde{I}_t^{(m)}$ and $\tilde{I}_t^{(m')}$ are defined similarly to $I_t^{(n)}$, but now the charges do not have to sum to zero, and the integrals over t_i and s_i must also include the contribution from R . It is straightforward to show that the combinatorics in the equation for I_t work out properly. For example, suppose there are m_+ positive charges in the multipole around t and m'_+ positive charges in the multipole around s . In the n th term for S , there are a total of n_+ positive inserted charges. We must sum over all the ways to pick the m_+ positive charges that are around t from the original n_+ positive charges, so the combinatorial factor becomes

$\frac{1}{n_+!} \binom{n_+}{m_+} = \frac{1}{m_+! m_+'!}$. Thus, the factorial of the number of positive charges in S is replaced by the product of the factorials of positive charges in each I_t . The negative charges work similarly.

Because $I_t^{(m)}$ and $I_t^{(m')}$ are independent of t and s , and we are left with evaluating the integral

$$R(\omega + Q\omega_0) = \int_{-\infty}^{\infty} d(t-s) R(t, t_1, \dots, t_m, s, s_1, \dots, s_m) e^{i(\omega + Q\omega_0)(t-s)}. \quad (73)$$

First, we will assume the multipoles around t and s are both neutral. In that case, when R is expanded out in powers of $1/(t-s)$, the first two terms are given by

$$R(t, t_1, \dots, t_m, s, s_1, \dots, s_m') = 1 + \frac{1}{(t-s)^2} 2g \sum_{i,j} q_i t_i p_j s_j. \quad (74)$$

(We note that when the regulators are carefully kept track of, for each term in the sum the $1/(t-s)^2$ may be regulated differently. However, since we only want to find the leading singular behavior of the integral over $t-s$, this short-distance behavior does not matter.) Performing the integral over $t-s$, we find that the singularity at $\omega = 0$ goes as

$$R_{\text{sing}}(\omega) = -\pi|\omega| 2g \sum_{i,j} t_i q_i s_j p_j. \quad (75)$$

If, instead, the multipoles around t and s are not neutral, but have net charge Q , then $R(\omega + Q\omega_0) \propto 1/(t-s)^{2Q^2g}$. This will give a singularity in S that goes as $|\omega \pm Q\omega_0|^{2Q^2g-1}$. Because this is smooth near $\omega = 0$ when $Q \neq 0$, this will not give any contribution to the $\omega = 0$ singularity.

Thus, the only contribution to the singularity at $\omega = 0$ comes from $R_{\text{sing}}(\omega)$. From the equation for I_t , we note that $\sum_{i=1}^{2n} t_i q_i I_t^{(n)} = \frac{1}{i} \frac{d}{d\omega_0} I_t^n$, where the t_i are understood to be inside the integral sign. Using this equation and the expressions for $S_t(\omega)$ and $R_{\text{sing}}(\omega)$, we find that the singular part of the noise near $\omega = 0$ is

$$S_t^{\text{sing}}(\omega) = 4\pi g \left(\frac{d}{d\omega_0} I_t \right)^2 |\omega|. \quad (76)$$

Now we can write the expression above in terms of the differential conductance $G_t = dI_t/dV$ and the Hall conductance $G_H = \nu e^2/h$, and set $\omega_0 = e^*V/\hbar$ to obtain

$$S_t^{\text{sing}}(\omega) = 2 \hbar |\omega| \frac{G_t^2}{G_H} \quad (77)$$

If we divide by $S^{(0)}(\omega) = \frac{\nu}{2\pi} |\omega| = \hbar |\omega| G_H$, and write $g_t = G_t/G_H$, we find that the normalized slope of the tunneling noise is

$$A_t = \frac{S_t^{\text{sing}}(\omega)}{S^{(0)}(\omega)} = 2g_t^2, \quad (78)$$

as we conjectured in Section III.

VI. CONCLUSION

In this paper we present results on the low frequency part of the noise spectrum for tunneling currents between fractional quantum Hall edge states. In addition to corrections to the classical shot noise level $S = 2e^*I$ due to generalized statistics of the quasiparticles, the low frequency noise is not white or frequency independent. There is cusp $\propto |\omega|$ in the spectrum, and we show in this paper that the proportionality constant depends directly on the differential conductance. Any anomalous scaling behavior on voltage V or tunneling amplitude Γ are all

contained in the implicit dependence of the differential conductance on V, Γ .

In addition to studying the noise in tunneling and transmission currents, we study the auto and cross-correlations of density-density fluctuations between pairs of terminals in a four probe measurement scheme. By looking into this larger class of correlations we can identify different transformation laws under the duality symmetry existent in the tunneling problem. More specifically, we can identify (see Sections II and III) correlations that transform as either “odd” or “anti-symmetric”, or “even” or “symmetric” quantities under duality. The

tunneling conductances is an example of an “odd” quantity.

The transformation laws under duality are suggestive of relations between the conductance and the slope of the noise correlations. We pursue these relations via an ansatz, which we support through a perturbative calculation. We show that the perturbative expansions for the slope of the noise spectrum at low frequencies matches term by term the perturbative expansion series for differential conductances (or squares of it). The equality makes use of the fact that the tunneling Hamiltonian is a cosine potential at the point contact. Therefore, our results are, in principle, particular to the boundary sine-Gordon model. The equalities are independent of the Luttinger parameter, so that one can perform the expansions around either the electron or the quasiparticle tunneling limits, obtaining the same relations to the differential conductances. The perturbative expansions around these dual points are valid in complementary regimes, so that the relations to the conductance hold for arbitrary coupling.

Recently, the issue of finite frequency correlations has been studied by Lesage and Saleur in Ref. [6] using the thermal Bethe ansatz and renormalized form factors. They also found that there is the $\propto |\omega|$ singularity, and calculated the prefactor. However, their V and Γ dependent prefactor does not coincide with the square of the differential tunneling conductance. The source of discrepancy remains unclear. Among the interesting and open questions are those related to 1) dressing of the reflection coefficients in the S -matrix 2) radius of convergence of the perturbative expansion. The first issue arises because the dressing used for calculating the conductance and the noise in Ref. [6] are different. The rationale for the choice is that only states near the rapidity edge (“Fermi” level) of the solitons should participate in the low ω noise. In the case of the conductance, all states under the “Fermi” sea contribute to the current. It is unclear if this distinction is valid, since for the differential conductance only states near the “Fermi” edge contribute. Of course, the system is interacting, and the level positions depend self-consistently on the occupation of other levels; but this rearrangement of levels should also be expected when considering excitations on the scale ω for the noise correlations. The second issue concerns the validity of the perturbative expansion to all orders. We believe the radius of convergence of the series is finite, just as in the calculation of the conductance [4,9]. Moreover, the region of validity of one expansion ends where the window for the dual expansion begins. Now, although there is the duality symmetry in the boundary sine-Gordon model, the theory in the infrared fixed point requires renormalization via neutral (density) counterterms [8]. At first, one may expect that these counterterms would spoil the relation between the slope of the noise spectrum and the conductance. However, we have checked that at least the counterterm found in Ref. [8] appear to leading order as ω^2 corrections, leaving the

linear ω term unaffected. The discrepancies between our results and those of Ref. [6] point to the need to better understand the dynamical and non-equilibrium, $\omega, V \neq 0$, aspects of the family of impurity problems described by the boundary sine-Gordon model, as well the differences between the Schwinger-Keldysh non-equilibrium formulation and the scattering approach [14].

Finally, we would like to put the contents of this paper in the context of the recent advances in understanding the dual descriptions of the boundary sine-Gordon problem [9,15,16]. An integral representation was found for the conductance, in which the dual expansions are controlled by the coupling dependent pole structure and branch cut structure of the integrand. One should expect that the slopes of the noise spectrum should have a similar integral representation. Moreover, the different transformation laws (such as the “odd” and “even” cases introduced here) should be manifest in the form of the integral representations, and are natural quantities for future studies.

ACKNOWLEDGEMENTS

We would like to acknowledge many interesting discussions with Hubert Saleur, who sent us the preprint of Ref. [6] prior to submission. Our work was motivated by these discussions, in an effort to better understand the connection between the duality relations and perturbative identities to the Bethe ansatz solution and the form factors. We would also like to thank Xiao-Gang Wen for many enlightening discussions. D.F. would like to acknowledge the Mary Ingraham Bunting Institute which provided the opportunity to start this project.

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